IX More classification results

A. The Farey graph

we will need to keep track of curves on tori

the Farey graph is a convenient way to do that

recall after fixing a basis λ, μ for $H_1(T^2)$ we can

represent any simple closed curve γ as

$$p\lambda + q\mu \quad \text{or} \quad \begin{pmatrix} p \\ q \end{pmatrix}$$

where p, q are relatively prime

and further as $q/p \in \mathbb{Q}^* = \mathbb{Q} \cup \{\infty\}$

so simple closed curves on $T^2 \leftrightarrow \mathbb{Q}^*$

the Farey graph lives in the unit disk $D^2 \subset \mathbb{R}^2$

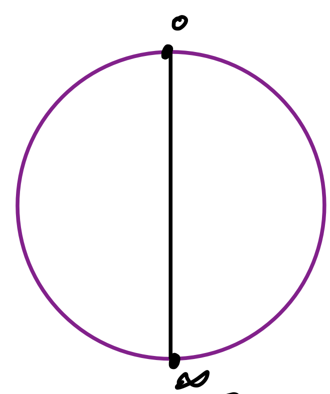
put the hyperbolic metric $\frac{4}{(1-r^2)} g_{\text{Euclidean}}$ (don't really need this, just use to connect points on ∂D^2 by geodesics)

we construct the Farey graph as follows

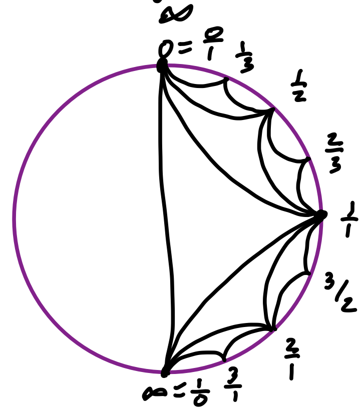
lable the point $(0,1)$ by $0 = \frac{0}{1}$

$(0,-1)$ by $\infty = \frac{1}{0}$

connect them with a hyperbolic geodesic



if p is a point on ∂D^2 with positive x-coord that is half way between labled point $\frac{a}{b}$ and $\frac{c}{d}$ label p with $\frac{a+c}{b+d}$ and

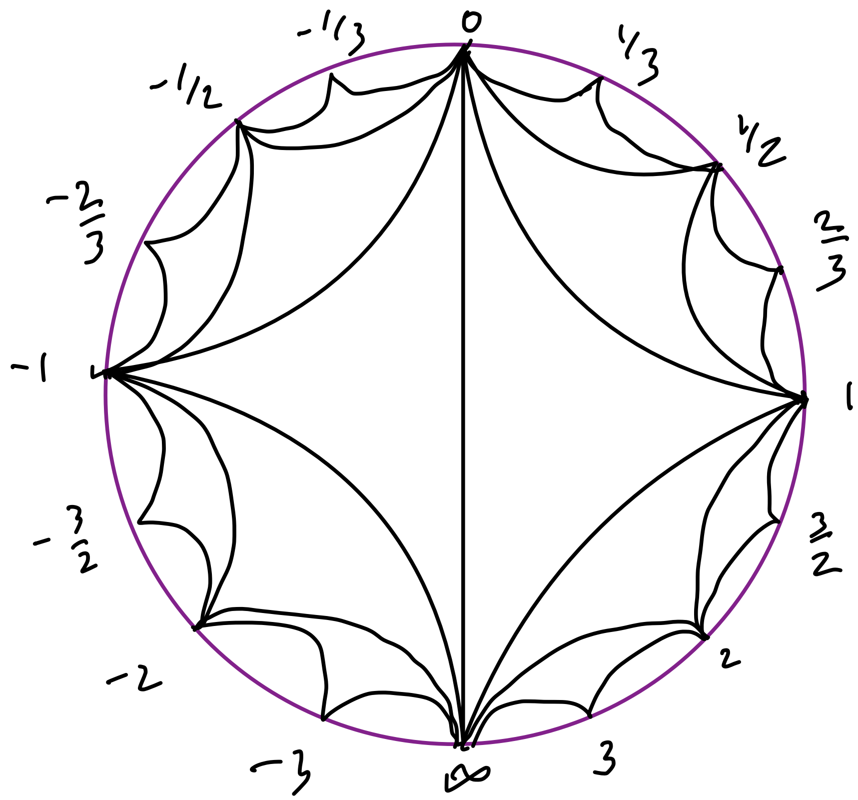


connect p to $\frac{a}{b}, \frac{c}{d}$ by a hyperbolic geodesic

iterate construction

now do same for $p \in \partial D^2$ with negative x-coord

but use $\infty = \frac{-1}{0}$ instead of $\frac{1}{0}$



exercise:

1) show all elements in $\mathbb{Q}^*$ show up as lables in the Farey graph and they are "in order" moving clockwise from $-\infty$ to ∞

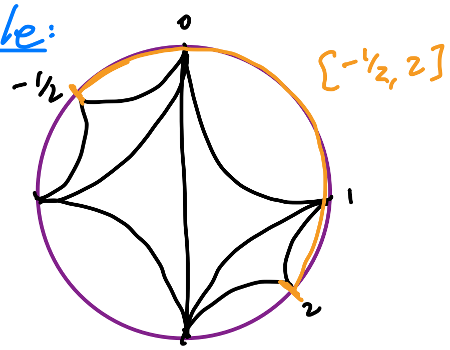
2) show two vertices correspond to curves that form a basis for $H_1(T^2)$ iff $\exists$ an edge between them in Farey graph

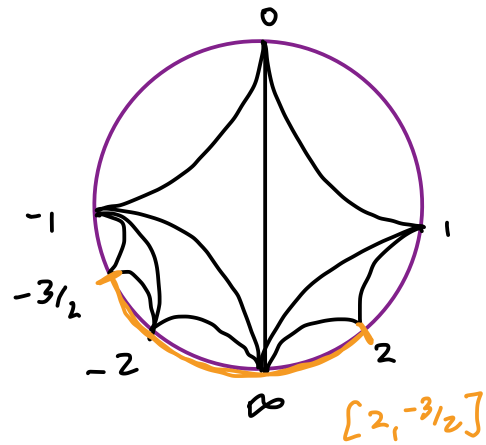
we let $[s_0, s_1]$ denote the vertices in the Farey graph that are clockwise of s_0 and anticlockwise of s_1

(including end points)

similarly for (s_0, s_1), $(s_0, s_1]$, and $[s_0, s_1)$

example:





B. Basic slices

a contact manifold $(T^2 \times [0,1], \xi)$ is called a basic slice if

1) ξ is tight
2) $T_i = T^2 \times \{i\}$ is convex with $\#\Gamma_{T_i} = 2$ for $i = 0, 1$
3) v_0, v_1 form an integral basis for $H_1(T^2 \times \{0\})$

   where v_i is a minimal length vector with slope equal to the slope s_i of Γ_{T_i} i.e. s_0, s_1 connected by an edge in Farey graph

4) the slope of the dividing curves on any convex torus T in $T^2 \times [0,1]$ is in $[s_0, s_1]$

   this condition is called minimally twisting

Th^m 1:

for each pair of slopes s_0, s_1 connected by an edge in the Farey graph there are exactly two basic slices with dividing slopes s_0 and s_1.
Moreover, their relative Euler classes are given by

$$\pm (v_1 - v_0) \in H_2(T^2 \times [0,1]) \cong H^2(T^2 \times [0,1], \partial(T^2 \times [0,1]))$$

where v_i is as in (3) above

Proof:

we prove theorem for $s_0 = \infty$ and $s_1 = -1$

the general case follows since there is a diffeo of $T^2 \times [0,1]$

taking any s_0, s_1 as in thm to $\infty, -1$

Thm clearly follows from

lemma 2: there are at most two basic slices with $s_0 = \infty$ and $s_1 = -1$

lemma 3: there are two basic slices with $s_0 = \infty$ and $s_1 = -1$ and they are distinguished by their relative Euler classes which are $\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \in H^2(T^2 \times [0,1], \partial) \cong H_1(T^2 \times [0,1])$

Proof of lemma 2:

let ξ be a basic slice with $s_0 = \infty$, $s_1 = -1$

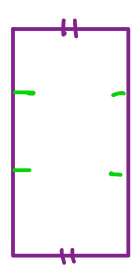
the characteristic fol^n on $\partial(T^2 \times [0,1])$ determines ξ near boundary

let T_i be a convex torus in an invariant neighborhood of $T^2 \times \{i\}$

we can assume the char fol^n on T_i is standard with ruling slope 0

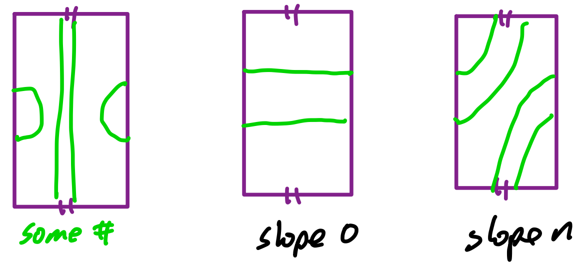
let A be an annulus running from T_0 to T_1 so that ∂A is a ruling curve on T_0 union one on T_1

we can make A convex (why!)

Γ_A near ∂A is  by lemma VII.11

by Giroux criteria only possibilities for Γ_A

are



or 2 curves of slope n

if we had first case then we could Legendrian realize (Th^m VII.7)

a slope 0 curve L on A

let T' be a torus isotopic to $\partial(T^2 \times [0,1])$

that contains L

we can make T' convex without moving L (why?)

now since $tw_\xi(L, T') = 0$ but must be $-\frac{1}{2}\#(L \cap \Gamma_{T'})$

we see L is disjoint from $\Gamma_{T'}$

so T' is a convex torus with slope $0 \notin [\infty, -1]$

⊗ minimal twisting so can't have this Γ_A

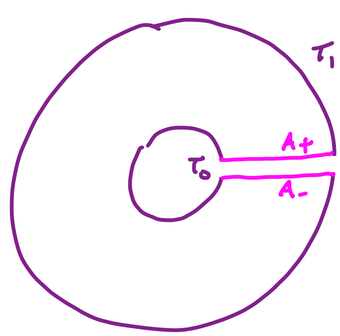
Claim:

$$\boxed{\text{In all other cases we can isotop } A \text{ so that } \Gamma_A \text{ has slope } 0}$$

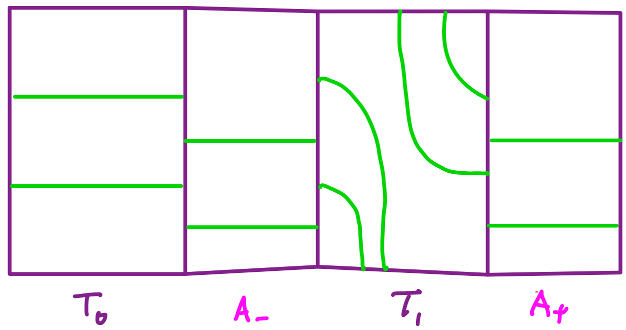
so ξ determined near A!

given this we can cut $T^2 \times [0,1]$ along A to get a solid torus S

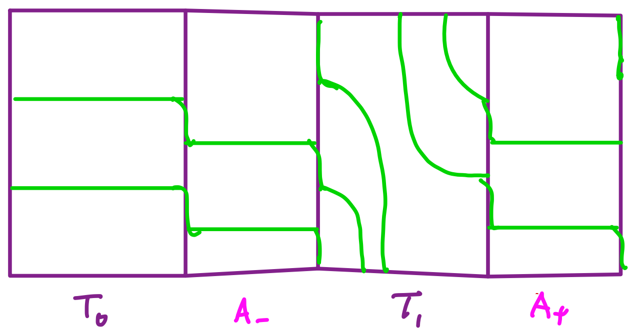
$([(T^2 \setminus \text{essential closed curve}) = \text{annulus}] \times I = S^1 \times D^2)$



what is Γ_S?



if we round corners as in lemma VII.12 we get



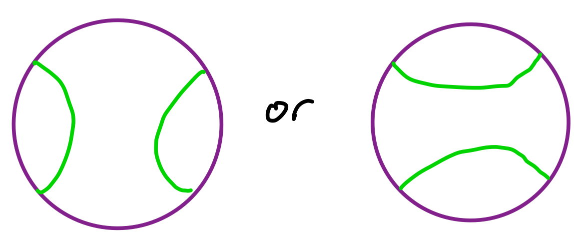
that is ∂S has 2 dividing curves of slope $-\frac{1}{2}$

we can isotop ∂S to be in standard form with dividing slope ∞

let D be a meridional disk for S with ∂D
a ruling curve

we can make D convex (why?)

by lemma VII.11 and Giroux criterion
we know Γ_D is



if we fix one of these then contact structure
is determined near D

note what is left is a 3-ball so Eliashberg's classification on B^3 determines ξ here

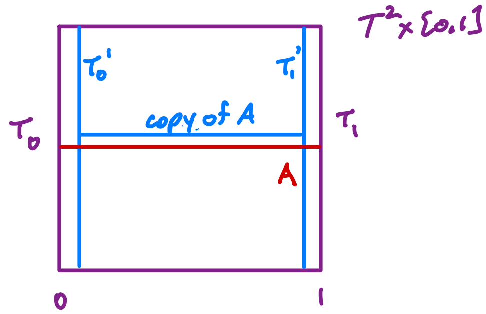
note: In the whole process above there was only one place where the contact structure was not completely determined by the initial data.

That was the dividing set on D and there were only two possibilities

$\therefore$ there are at most 2 basic slices !

Proof of Claim:

we isotop A as follows



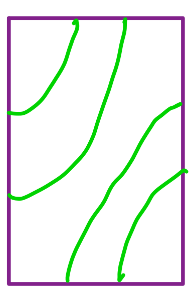
T_i' a copy of T_i pushed in along contact vector field

copy of A with ∂ ruling curves on $T_0' \cup T_1'$

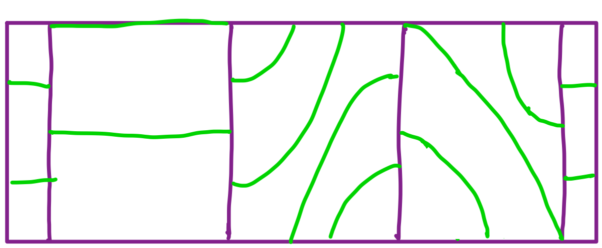
for A' as shown above and round corners

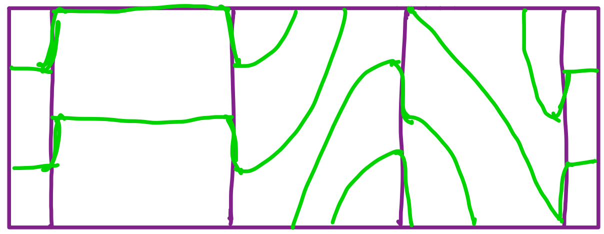
A' is smoothly isotopic to A and if

Γ_A was



then $\Gamma_{A'}$ is





exercise: why did outer 2 roundings go up instead of down like all others?

so slope went from -1 to 0

keep doing this can go from any neg. slope to 0

exercise: show that pushing A in the other direction will decrease the slope by 1 ▦

Proof of lemma 3:

consider $T^2_{(x,y)} \times \mathbb{R}_z$ with contact structure

$$\xi = \ker((\sin 2\pi z)\,dx + (\cos 2\pi z)\,dy)$$

notice when we pull ξ back to the universal cover $\mathbb{R}^3$ we get

a contact structure contactomorphic to the standard one

thus it, and ξ, are tight

consider $T^2 \times [0, 1/8]$

note: 1) $T^2 \times \{0\}$ has a linear fol^{n} of slope ∞

2) $T^2 \times \{1/8\}$ " " " " -1

as we did in the example just after Th^{m} VII.4 we can

C^∞ small perturb $T^2 \times \{0, 1/8\}$ so that they are convex with 2

dividing curves of slope ∞ and -1, respectively

denote this contact manifold by $(T^2 \times [0,1], \xi)$

and note it obviously satisfies all the properties of

being a basic slice except being minimally twisting

so we are left to show this

we need a lemma, but first some notation

let $M_{r,r'} = T^2\times[a,b]$ with contact structure ξ above

such that the slope of characteristic foliation

on $T^2\times\{a\}$ is r and

on $T^2\times\{b\}$ is r'

and $0<b-a<\frac{1}{2}$

note char fol^n on $T^2\times\{t\}$ is linear and

moving from r clockwise to r'

lemma 4:

$$\boxed{\begin{array}{l}\text{if } s\notin[r,r'] \text{ then there is no convex torus in } M_{r,r'} \\ \text{isotopic to } \partial M_{r,r'} \text{ with dividing slope } s \\ \text{(also no linear fol}^n\text{ of slope } s)\end{array}}$$

assume this for now and we finish the proof of lemma 3

recall our $(T^2\times[0,1],\xi)$ was obtained from $T^2\times[0,1/8]$

with $\ker(\sin(2\pi z)dx+\cos(2\pi z)dy)$ by perturbing

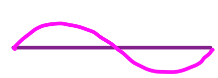
the boundary to be convex

we discuss this perturbation more carefully (focusing on $T^2\times\{0\}$

but same applies to $T^2\times\{1/8\}$)

$\exists$ a function $f: T^2\to[-\delta,\delta]$ such that the graph

of f in $T^2\times\mathbb{R}$ makes $T^2\times\{0\}$ convex



$\times S' = T^2\times\{0\}$

graph of f

let $f_t(p)=tf(p)$ for $t\in[0,1]$

let (M_t, ξ_t) be the contact manifold obtained from $T^2 \times [0, 1/8]$

by purturbing $T^2 \times \{0\}$ by f_t (and similarly

for $T^2 \times \{1/8\}$)

note: $M_t \subset T^2 \times [-t\delta, 1/8 + t\delta]$

from lemma 4 we know $T^2 \times [-t\delta, 1/8 + t\delta]$

contains no convex tori parallel to boundary

with slope outside $[1/\varepsilon_t, -1+\varepsilon_t]$ for some

$\varepsilon_t \to 0$ as $t \to 0$

note: each M_t is canonically (up to isotopy) $T^2 \times [0,1]$

we only use t to make clear which subset

of $T^2 \times \mathbb{R}$ we are talking about

Claim: $(M, \xi) = (M_1, \xi_1)$ is contactomorphic to a subset of

(M_t, ξ_t) for $t \in (0,1]$, by a contactomorphism that

does not change slopes on $T^2 \times \{s\}$

given this we see (M, ξ) is minimally twisting, since if not,

there is a torus T with slope s outside $[\infty, -1]$

$\therefore$ s is outside $[\frac{1}{\varepsilon_t}, -1+\varepsilon_t]$ for some t

but $(M, \xi) \subset (M_t, \xi_t) \subset T^2 \times [-t\delta, 1/8 + t\delta]$

this contradicts observation above!

Proof of Claim:

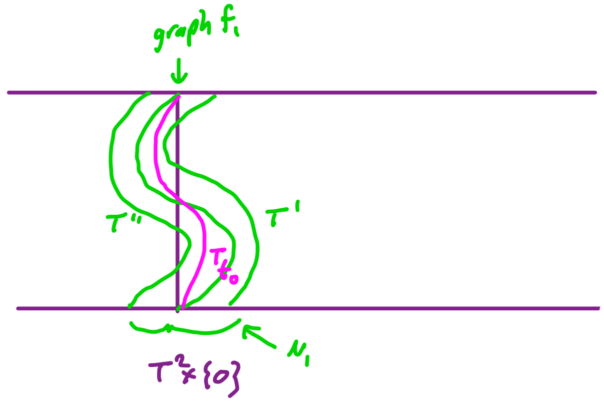
for each $t \in (0,1]$ there is some nbhd $N_t = T^2 \times [-\varepsilon, \varepsilon]$ of

graph of f_t on which ξ is $[-\varepsilon, \varepsilon]$-invariant let $O_t' = \{ s \in [0,1] \text{ s.t. graph of } f_s \subset \text{int } N_t \}$

O_t = open, connected subset of O_t' containing t

want to see for fixed $t_0 \in [0,1]$ can assume $(M_1, \xi_1) \subset (M_{t_0}, \xi_{t_0})$

if $t_0 \in O_1$ then note



let $T' \cup T'' = \partial N_1$

let M_1' be region in $T^2 \times \mathbb{R}$ bounded by T' and corresponding T' for $T^2 \times \{1/8\}$

M_1'' be same but for T''

<u>note:</u> M_1, M_1', M_1'' all contactomorphic!

and $M_1' \subset M_{t_0}$ as claimed, and $M_{t_0} \subset M_1''$

if $t_0 \notin O_1$ but $O_{t_0} \cap O_1 \neq \emptyset$ contains t_1 then from above

$M_1 \subset M_{t_1} \subset M_{t_0}$

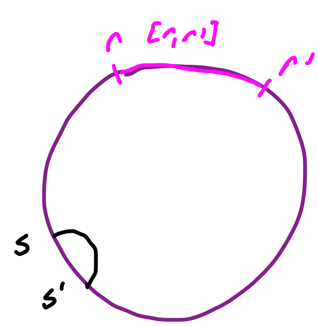
<u>exercise:</u> do general case

<u>Proof of lemma 4:</u>

suppose there were such a torus T

<u>exercise:</u> there is a slope s' such that $s' < s < r < r'$

<u>Hint:</u> consider Farey graph



exercise: there is a diffeomorphism ϕ of T^2 sending

s to ∞ and s' to 0

and ϕ sends r, r' to negative integers

extend ϕ to a diffeo $T^2 \times [a,b]$ to itseld by id on $[a,b]$

exercise: show $\phi_* \xi$ is a subset of the contact structure

$\ker(\sin(2\pi z)dx + \cos(2\pi z)dy)$ on $T^2 \times (0, 1/4)$

exercise: $(T^2 \times (0, 1/4), \ker(\sin(2\pi z)dx + \cos(2\pi z)dy))$

embeds in (S^3, ξ_{std})

Hint: think of S^3 as unit sphere in $\mathbb{C}^2$

$H = \{z_1 = 0\} \cup \{z_2 = 0\}$ is a Hopf link in S^3

$S^3 - H = T^2 \times (0,1)$ and contact planes

tangent to $(0,1)$-factor and induce

linear fol^n on $T^2 \times \{t\}$ with slope

in $(\infty, 0)$ and increasing as t goes

from 0 to 1

$\therefore$ $\phi(M_{r,r'}) \subset S^3$ as a nbhd of a Heegaard

torus and $\phi(T)$ is a convex torus with

dividing slope 0

a Legendrian divide on $\phi(T)$ bounds a disk

$\therefore$ $tb(\text{this unknot}) = 0$ $\otimes$ tightness

$\therefore$ T does not exist